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FRICTION AND THE BOWED STRING*

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Summary

Some aspects of bowed string vibration are discussed with particular emphasis on a regime which is sensitive to details of the friction–velocity characteristics. Apart from its intrinsic interest this study may shed light on the behaviour of other friction-driven oscillators since the rather successful theoretical modelling developed for the bowed string may be applicable to other problems. Conversely, bowed string studies may be able to benefit from the interaction with other work on friction.

1. Introduction

The bowed string is perhaps unique among frictional oscillators in that the oscillations are actively sought, rather than being an undesirable thing to be avoided. This engenders a particular attitude to theoretical modelling. If one is modelling a phenomenon only in order to find ways of preventing it, one may not be very concerned about details of the behaviour. Players of bowed string instruments, however, are very interested in the fine details of the string motion and in how these can be controlled to provide a variety of musical expression. Many of these details are now understood quite well as a result of a long history of theoretical and experimental study of the problem. This makes the bowed string probably the most thoroughly understood frictionally driven oscillator and, while much of this work exploits special features of the problem, it seems probable that some of the insights gained will have applications elsewhere. Space does not permit a comprehensive review of the subject here and we concentrate on some aspects likely to be of wider significance.

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2. Model formulation

We first derive the basic equation describing the problem under the simplest idealizations. We suppose that the string is bowed at a single point along its length with constant bow speed v_b and normal force f_b. At the bowed point let the string's transverse velocity be $v(t)$ and let the frictional force be $f(t)$, where t is the time. These two quantities are related in two distinct ways. First, we shall suppose that the frictional force has a functional dependence on the relative velocity of the bow and string of the type indicated by the heavy curve in Fig. 1. Secondly, $f(t)$ and $v(t)$ are connected

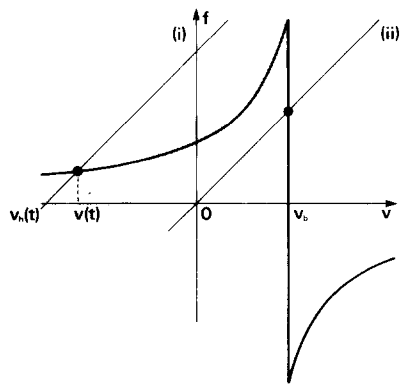

Fig. 1. Heavy full curve: a typical non-linear characteristic $f(v)$ curve for friction in a bowed string as usually idealized. The sloping straight lines show a graphical construction to solve eqn. (3): (i) at time t for a general model; (ii) at all times for an infinitely long string. If the sloping line falls in the shaded region it has more than one intersection with the heavy curve and hysteresis occurs as described in the text.

by a complicated linear dissipative system, the string and its terminations. Supposing this system to have an impulse response function $g(t)$, we are immediately led to the non-linear integral equation

$$v(t) = \int_0^\infty g(t')f\{v(t-t')\}\,dt' \tag{1}$$

Equation (1) is by no means special to the string; it describes any system driven by friction (modelled as a functional f–v relation) applied at a point. Different linear systems are characterized by different impulse response functions $g(t)$.

Our first concern is with the behaviour of $g(t)$ for very small times. The string, in common with many other simple wave-bearing systems which do not have a finite mass concentrated at the drive point, has an initial Dirac delta function contribution and it is convenient to separate this from the remainder of $g(t)$ by writing (see ref. 1 for further discussion)

$$g(t) = \tfrac{1}{2} Y\delta(t) + g_h(t) \tag{2}$$

where Y is the characteristic admittance of the string and $g_h(t)$ is finite in the vicinity of $t = 0$ (and of course zero for $t < 0$). Equation (1) then yields

$$v(t) = \tfrac{1}{2} Yf\{v(t)\} + v_h(t) \tag{3}$$

where

$$v_h(t) = \int_0^\infty g_h(t')f\{v(t - t')\}\, dt' \tag{4}$$

The function $v_h(t)$ depends only on the past history of the motion; it represents the net effect of reflections arriving back at the bowed point at time t.

If the past history, and therefore $v_h(t)$, is known then eqn. (3) determines $v(t)$ and $f(t)$ by the graphical construction shown in Fig. 1: they are given by the intersection of a straight line of slope $2/Y$ and intercept $v_h(t)$ (*e.g.* the line labelled (i) in Fig. 1) with the friction characteristic $f(v)$. We now see that the vertical (sticking) portion of the characteristic (heavy curve) presents no problem for this model: the intersection is perfectly well-defined (for example with the straight line labelled (ii)). A problem can arise, however, if the shape of the $f(v)$ curve allows a region like the one shaded in Fig. 1. If $v_h(t)$ lies in the range of the v axis within such a shaded region there is an ambiguity in the determination of v and f from v_h since the line then intersects the curve at three points rather than one. The resolution of this ambiguity has been shown to be the one which might have been expected, a hysteresis cycle involving the outer two of the three intersections [1]. The middle intersection is unstable and never occurs. This hysteretical behaviour has important observable consequences to which we return later.

Before considering the influence of $g_h(t)$ on the motion of the string, we note in passing that we have already solved the problem in one limiting case. For an infinitely long string, or one with such high damping that no reflections return with significant strength, we have $v_h = 0$ at all times. Thus bowing an infinitely long string cannot make it oscillate; the string simply moves steadily in the way determined in Fig. 1 with $v_h = 0$ (the sloping line labelled (ii) in Fig. 1). Whether it sticks (as here) or slips depends upon the shape of the $f(v)$ curve and the value of the bow speed v_b. Only if $v_h = 0$ lies in the ambiguous region can anything remotely interesting happen: with a choice of two stable states available it might be possible for the string to be switched occasionally between sticking and slipping by perturbations, for example, arising from spatial irregularities in the rosin on the bow.

If a finite string is subjected to a transitory disturbance in $f(t)$ then it will eventually come to rest in its original position. This means that the integral over all time of the function $g(t)$ must vanish. The integral of $g_h(t)$ over all time thus has the value $-Y/2$. During a steady oscillation $f(t)$ has a positive mean value (for otherwise the $f(v)$ of Fig. 1, and the fact that v must be less than v_b for at least part of the cycle, would imply a negative mean rate of working) and so it follows, by taking the time average of eqn. (4) and reversing the order of integration, that $v_h(t)$ has a negative mean value. This argument applies, of course, to many other physical systems as well.

3. Raman's classification

For the next stage of the argument we use a special feature of the behaviour of stretched strings. Waves on a string travel approximately non-dispersively, so that in modal terms the natural resonance frequencies are approximately harmonically spaced. This feature is responsible for the most remarkable feature of the string as a frictional oscillator, its pitch stability. The vibration period of most stick–slip oscillators (squeaking doors, for example) varies strongly with the imposed conditions such as normal force and speed of relative movement. If that were true of the bowed string the violin would hardly be a viable musical instrument. However, while sufficiently drastic changes in bowing parameters can produce large changes of pitch (or indeed raucous, more-or-less pitchless, sounds), there are usefully wide ranges of these parameters in which the pitch is very nearly constant.

This constant pitch closely matches that of the string when plucked and this observation provided the key to the earliest extensive theoretical study of the vibration regimes of the bowed string by Raman [2]. The essence of his argument was that if the result of frictional driving is to produce a periodic oscillation with all harmonic components close to resonances of the free highly resonant system then the friction force $f(t)$ must be more-or-less constant throughout the cycle. If it were not, the variations in force with the string's natural frequencies would evoke a resonant response rather than a steady vibration. (Of course, on a real lossy string some force fluctuations are needed to compensate for dissipation but Raman's argument is evidently a good first approximation.) For a friction characteristic having the general shape shown in Fig. 1 a given constant force allows $v(t)$ to take just two values, one slipping and one sticking. Thus to a first approximation the motion must consist of an alternation between these two velocities in some pattern. This led Raman to an elegant kinematic argument which allowed him to classify the many different non-linear vibration regimes observed in the laboratory in terms of the number of "corners" or velocity jumps travelling back and forth along the string. Each of these jumps has the same magnitude given by the difference of the two allowed velocities.

The simplest such motion is actually a good approximation to the regime which violinists usually want. This regime was first described by Helmholtz [3] and is illustrated in Fig. 2. A single corner shuttles around the visible envelope of the string's vibration triggering transitions between sticking and slipping each time it passes the bow. There is thus one sticking and one

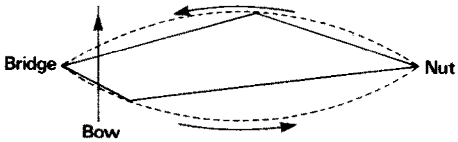

Fig. 2. The simplest motion of a bowed string first observed by Helmholtz. The transverse scale is exaggerated. The visible envelope of the motion is shown as a broken curve together with two "snapshots" of the moving string at different instants. The position and direction of the bow are indicated and also the directions of travel of the Helmholtz corner. The string is sticking to the bow at both the instants shown.

slipping period per cycle. The timekeeping by the Helmholtz corner is responsible for the pitch stability mentioned above. It is now clear why the non-dispersive nature of wave propagation on a string is important. If the corner "spreads" too much during its travel it will become ineffective at triggering precisely timed transitions between sticking and slipping. An important area of research into the bowed string is to map out the region of the player's parameter space in which the Helmholtz motion, or some approximation to it, can be obtained. Space permits no discussion here but see refs. 4 - 6.

Raman's argument for deducing the character of the motion in the many observed regimes does not seem very closely tied to the integral equation formulation of the problem given above. For the present, we can merely note that the particular forms of $g_h(t)$ for a range of fairly realistic string models allows a very efficient numerical simulation of the dynamical system represented by eqn. (3) [1, 7] and that such simulations confirm Raman's classification as a good first approximation. They also confirm a second line of enquiry, initiated by Raman, in which string dissipation was modelled in an idealized way [4, 5].

An interesting issue arises here concerning which string models to use. It turns out to be vital not to use the usual "textbook" model of a lossless rigidly anchored perfect string since that model gives very unrealistic results [7 - 10] — principal among them the fact that all periodic solutions are unstable! The more realistic models used in our simulation studies, alternatively, seem able to reproduce nearly all the known features of the behaviour of real bowed strings. It turns out that the playability of real strings depends crucially on their torsional degree of freedom [7, 10].

4. Hysteresis and the flattening effect

We end this brief survey of work on the bowed string with a discussion of an interesting phenomenon which arises when we ask what effect the hysteresis described in Section 2 has on the Helmholtz motion in a realistic string. This phenomenon is interesting in the present context because it depends critically on some details of the frictional behaviour whereas the kinematic argument used by Raman treats friction merely by seeking free motions of an ideal string which are not inconsistent with the presence of friction at the bowed point.

On a real string there will be some rounding of the Helmholtz "corner" during its travel along the string. The rounding is due to both dispersive and dissipative effects in the string itself and to reflection processes at the ends of the string. This corner-rounding effect can be represented in the model by the convolution of the outgoing velocity waveform with some kind of hump-like function to give the (delayed) incoming velocity wave at the bow. (This approach lies at the heart of the efficient simulation algorithm mentioned above).

We now consider the interaction of a rounded Helmholtz corner with friction at the bow during the processes of release and capture. Detailed views of the waveforms of $v_h(t)$ and $v(t)$ during these processes are shown in Fig. 3(a), with hysteresis in operation. It will be seen that the interaction implied in Fig. 1 produces "corner sharpening" which for a steady state oscillation must balance the corner-rounding effect just described. The details of that balance determine the oscillation period and waveform $v(t)$. The steady vibration waveform of $v(t)$ incorporating the interactions of Fig. 3(a) is shown in Fig. 3(b). The waveform was obtained for a particularly simple (though rather unrealistic) simulation of bowing at the midpoint of a string with symmetric terminations. Figure 3 is taken from ref. 7 and is discussed in much more detail there.

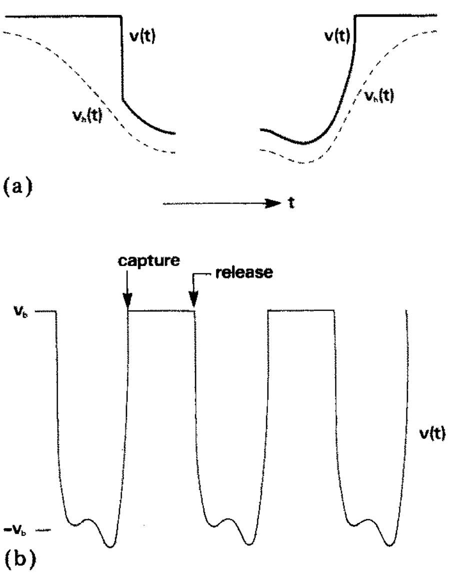

Fig. 3. Velocity waveforms from a simple simulation of a bowed string showing (a) waveforms of $v(t)$ (full curve) and $v_h(t)$ (broken curve) during the processes of release and capture and (b) the resulting steady oscillation waveform of $v(t)$. The model used for this simulation has a symmetrically terminated string bowed at its midpoint, details are given in ref. 7. The period of oscillation here is longer than the string's natural period by some 5%, *i.e.* nearly a semitone of flattening.

The oscillation regime illustrated in Fig. 3(b) is controlled by the combination of corner sharpening at the bow, corner rounding during travel to one end of the string and back and hysteresis. The essential factor to note is that hysteresis makes the processes of capture and release asymmetrical. This happens in such a way that there is a net delay in the round trip time of the Helmholtz corner during a steady state oscillation. In other words when hysteresis is significant the note plays flat. The magnitude of the effect depends on the strength of the corner-rounding processes and on the width of the ambiguous (shaded) region in Fig. 1 which will increase with increasing f_b.

The flattening effect is readily demonstrated on a violin as the normal force between bow and string is increased (use a low bow speed and play a high G string note for the strongest effect). It should be noted that such flattening is quite counter intuitive; as the force is increased, one might expect the pitch to rise, not fall, either because the string tension is slightly increased or because of heterodyning effects of the kind extensively discussed by Benade in the context of brass instruments [11] (since the overtones of the string will be systematically higher than those of a harmonic series because of finite bending stiffness). Nevertheless, the pitch falls and hysteresis explains the observation and so this is presumably a real effect.

The precise magnitude of the flattening, as well as the details of some other vibration regimes which we do not have space to discuss [6, 10], depend strongly on the shape of the friction characteristic. Thus for complete understanding we need to know $f(v)$. This requires measurements and an understanding of the physics of friction mediated by rosin. Some steady sliding measurements have been made [5], suggesting a curve like that sketched in Fig. 1. However, these measurements perhaps beg the question of the extent to which friction can be described simply by a single function, *e.g.* $f(v)$. More work is needed in this area. One piece of information is available immediately. One may readily verify that it is almost impossible to draw a bow across a string without vibration starting. It appears that the state of steady slipping is unstable. In the context of the simple friction curve model this is easily shown to imply that the slipping portion of the friction curve has a positive slope everywhere, as indicated in Fig. 1 [8, 9]. The instability must still apply to more sophisticated models of friction which might prove necessary. Ideally, what is wanted are methods of accurately measuring friction in a highly transient dynamical context. Some ideas on this subject have been floated [10] but little work has yet been done to our knowledge.

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